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Existence, Uniqueness and Comparison Results for Nonlinear Boundary Value Problems Involving a Deviating Argument

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1. INTRODUCTION AND PRELIMINARIES

In this paper we present existence, uniqueness and solution estimates for the differential equation with deviating argument

$$Lx(t) = f(t, x(t), x(\phi(t))) \tag{1}$$

defined on the interior $\dot{I}$ of the interval $I = [t_0, t_1]$.

We assume that all functions herein are real valued, $f: I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and ϕ is continuous on some interval containing I.

The prescribed boundary conditions for Eq. (1) are of the general Dirichlet/Neumann form

$$\begin{aligned} \alpha_0 x(t) - \alpha_1 x'(t) &= \alpha(t), \qquad t \in I^- = (-\infty, t_0], \\ \beta_0 x(t) + \beta_1 x'(t) &= \beta(t), \qquad t \in I^+ = [t_1, \infty), \end{aligned} \tag{2}$$

where α_i and β_i, $i = 1, 2$, are nonnegative reals satisfying $\alpha_0 + \alpha_1 > 0$ and $\beta_0 + \beta_1 > 0$. The functions $\alpha: I^- \to \mathbb{R}$ and $\beta: I^+ \to \mathbb{R}$ are assumed to be continuous.

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The operator $L\colon C^2(\dot{I}) \to C(\dot{I})$ has the form $Lx = -x'' + qx' + rx$, where

(i) the functions q and r are continuous on some interval containing I,

(ii) $r(t) \geqslant 0$ on I if $\alpha_0 + \beta_0 > 0$,

(iii) $r(t) > 0$ on I if $\alpha_0 + \beta_0 = 0$.

By a solution of the boundary value problem (1), (2) we mean a function $x \in C^j(\mathbb{R}) \cap C^2(\dot{I})$ which satisfies Eqs. (1) and (2). The index $j = 1$ if both $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. Otherwise, $j = 0$. Clearly any solution of the BVP (1), (2) is bounded on I.

In the special case when f is a function of t alone, it follows easily from the maximum principle [1, p. 13] that problem (1), (2) has at most one solution which is defined on I and which is in the class $C^2(\dot{I})$. Moreover, the existence of a $C^2(I)$ solution is guaranteed (cf. [2]) by the above regularity and boundary conditions. Using the boundary conditions this $C^2(I)$ solution can be extended to the whole of $\mathbb{R}$. As a consequence, there exists a unique solution of (1), (2) in $C^j(\mathbb{R}) \cap C^2(\dot{I})$. When f is the zero function we denote this solution by $z(\cdot, \alpha, \beta)$.

Employing the Green's function and the superposition principle for ordinary linear differential equations, the unique solution x of (1), (2) can be expressed in the form

$$x(t) = z(t, \alpha, \beta) + Ff(t), \tag{3}$$

where $F\colon C(\mathbb{R}) \to C^j(\mathbb{R}) \cap C^2(\dot{I})$ is defined by

$$\begin{aligned} Ff(t) = \xi(t) &= \int_{t_0}^{t_1} k(t,s) f(s)\, ds, && t \in I, \\ &= e^{\alpha_0(t-t_0)/\alpha_1} \xi(t_0), && t \in I^-, \alpha_1 \neq 0, \\ &= 0, && t \in I^-, \alpha_1 = 0, \\ &= e^{-\beta_0(t-t_1)/\beta_1} \xi(t_1), && t \in I^+, \beta_1 \neq 0, \\ &= 0, && t \in I^+, \beta_1 = 0, \end{aligned} \tag{4}$$

where $k(t, s)$ is the Green's function associated with the operator L and the homogeneous boundary conditions

$$\alpha_0 \xi(t_0) - \alpha_1 \xi'(t_0) = 0, \qquad \beta_0 \xi(t_1) + \beta_1 \xi'(t_1) = 0. \tag{5}$$

It follows from the maximum principle that $k(t, s)$ is positive on $\dot{I} \times \dot{I}$.

From these preliminaries we arrive at the equivalence of the nonlinear boundary value problem (1), (2) and the fixed point equation

$$x = c + Kx, \tag{6}$$

where $c = z(\cdot, \alpha, \beta)$ and $K: C(\mathbb{R}) \to C(\mathbb{R})$ is defined by

$$Kx(t) = Ff(t, x(t), x(\phi(t))), \qquad t \in \mathbb{R}. \tag{7}$$

In what follows we first derive some results for Eq. (6) and then apply them to the class of problem (1), (2).

2. Auxiliary Theorems

Let $\mathbb{R}_+$ denote the nonnegative reals and let $C_+(\mathbb{R})$ denote the space of all continuous functions from $\mathbb{R}$ into $\mathbb{R}_+$ endowed with the compact open topology and with the partial ordering $\leqslant$ defined by

$$u \leqslant v \qquad \text{if and only if} \quad u(t) \leqslant v(t) \qquad \text{for all } t \in \mathbb{R}.$$

We also assume that the spaces $C(\mathbb{R})$, $C(I^-)$ and $C(I^+)$ are endowed with the compact open topology.

Let θ denote the zero function of $C(\mathbb{R})$. A mapping Q from a segment $S = \{\theta \leqslant u \leqslant b\}$ of $C_+(\mathbb{R})$ into itself is called *increasing* if $u \leqslant v$ implies $Qu \leqslant Qv$, *order continuous* if $\lim_{n\to\infty} Qu_n = Qu$ whenever (u_n) is a monotone (i.e., decreasing or increasing) sequence in S converging to $u \in S$, and *order compact* if (Qu_n) has a convergent subsequence whenever (u_n) is a monotone sequence in S.

For a given $v \in C_+(\mathbb{R})$ the pair $x, \bar{x} \in C(\mathbb{R})$ is said to have the finite connecting *v-chain* of functions $y_i \in C(\mathbb{R})$, $i = 0, 1, \ldots, m$, if $y_0 = x$, $y_m = \bar{x}$ and $|y_i - y_{i-1}| \leqslant v$, $i = 1, 2, \ldots, m$, where $|\ |: C(\mathbb{R}) \to C_+(\mathbb{R})$ is defined by $|x|(t) = |x(t)|$, $t \in \mathbb{R}$. A mapping $K: C(\mathbb{R}) \to C(\mathbb{R})$ is said to be *v-chainable* if for each pair $x, \bar{x} \in C(\mathbb{R})$ the values Kx and $K\bar{x}$ have a finite connecting v-chain in $C(\mathbb{R})$. For given $x_0 \in C(\mathbb{R})$ and $b \in C_+(\mathbb{R})$ we denote

$$B_b(x_0) = \{y \in C(\mathbb{R}): |y - x_0| \leqslant b\}.$$

THEOREM 1. *Let $v, b \in C_+(\mathbb{R})$ and $K: C(\mathbb{R}) \to C(\mathbb{R})$ satisfy*

$$|Kx - K\bar{x}| \leqslant Q(|x - \bar{x}|) \text{ whenever } x, \bar{x} \in C(\mathbb{R}), \qquad |x - \bar{x}| \leqslant b, \tag{8}$$

where $Q: \{\theta \leqslant u \leqslant b\} \to \{\theta \leqslant u \leqslant b\}$ is an increasing, order continuous and order compact mapping for which θ is the only fixed point and $v + Qb \leqslant b$. Let K be v-chainable also. Then, for each $c \in C(\mathbb{R})$, the successive approximations $x_0 \in C(\mathbb{R})$, $x_{n+1} = c + Kx_n$, $n = 0, 1, \ldots$, converge to a unique solution $x(\cdot, c)$ of Eq. (6) in $C(\mathbb{R})$. Moreover, if $c, \bar{c} \in C(\mathbb{R})$ with $|\bar{c} - c| \leqslant v$, then we have the estimate

$$|x(\cdot, \bar{c}) - x(\cdot, c)| \leqslant u_*, \tag{9}$$

where u_* is the minimal solution of the equation

$$|\bar{c} - c| + Qu = u. \tag{10}$$

Proof. Let $c, x_0 \in C(\mathbb{R})$ be given, and denote $T = c + K$. Then T is v-chainable and the existence, uniqueness and convergence assertions are equivalent to the convergence of the iterations $T^n x_0$ to a unique fixed point x of T.

Assume first that T has a fixed point x. To prove the convergence $T^n x_0 \to x$, choose a finite v-chain $\{y_0, ..., y_m\}$ connecting Tx_0 and Tx. Since

$$|y_i - y_{i-1}| \leqslant v \leqslant b \qquad \text{for} \quad i = 1, ..., m,$$

the monotonicity of Q and condition (8) imply by induction that

$$|T^n y_i - T^n y_{i-1}| \leqslant Q^n b \qquad \text{for} \quad n = 0, 1, ..., \text{ and } i = 1, ..., m.$$

Thus

$$\begin{aligned} |x - T^{n+1} x_0| &= |T^n(Tx) - T^n(Tx_0)| \\ &\leqslant \sum_{i=1}^{m} |T^n y_i - T^n y_{i-1}| \leqslant m Q^n b, \end{aligned}$$

which implies the convergence $T^n x_0 \to x$, since the hypotheses given for Q ensure that $Q^n b \to \theta$ in $C_+(\mathbb{R})$. If $\bar{x} \in C(\mathbb{R})$ is also a fixed point of T, then by choosing $x_0 = \bar{x}$ above and noting that $T^n \bar{x} = \bar{x}$ for all $n = 0, 1, ...$, it follows that $x = \bar{x}$, which proves the uniqueness assertion.

To show that $T = c + K$ has a fixed point, assume for the moment that

$$|x_0 - c - Kx_0| \leqslant v. \tag{a}$$

Then, for each $y \in B_b(x_0)$

$$\begin{aligned} |x_0 - Ty| &\leqslant |x_0 - Tx_0| + |Tx_0 - Ty| \\ &\leqslant v + Q|y - x_0| \leqslant v + Qb \leqslant b, \end{aligned}$$

so that T maps the ball $B_b(x_0)$ into itself. Thus

$$|T^k x_0 - x_0| \leqslant b \qquad \text{for all } k = 0, 1, ...,$$

which, together with (8), implies by induction that

$$|T^{n+k} x_0 - T^n x_0| \leqslant Q^n b, \qquad n, k = 0, 1, ... \tag{b}$$

Since $Q^n b \to \theta$ in $C_+(\mathbb{R})$, it follows that $(T^n x_0)$ converges in $C(\mathbb{R})$. Denoting $x = \lim_{n\to\infty} T^n x_0$, we obtain from (b), as $k \to \infty$, that

$$|x - T^n x_0| \leqslant Q^n b, \qquad n = 0, 1, \ldots,$$

whence

$$|x - Tx| \leqslant |x - T^n x_0| + |T^n x_0 - Tx| \leqslant Q^n b + Q^n b \to \theta.$$

Thus $x = Tx$, so that T has a fixed point, provided that (a) holds. To remove this auxiliary condition, choose $x_1 = Tc = c + Kc$ and denote $c_0 = x_1 - Kx_1$. Since $c - c_0 = Kx_1 - Kc$ and K is v-chainable, there exists a v-chain $c_0, \ldots, c_m$ from c_0 to c. Then

$$|x_1 - c_1 - Kx_1| = |c_0 - c_1| \leqslant v,$$

whence condition (a) holds when $c = c_1$ and $x_0 = x_1$. By the above proof $T_1 = c_1 + K$ has a fixed point x_2. This in turn satisfies

$$|x_2 - c_2 - Kx_2| \leqslant v,$$

so that $T_2 = c_2 + K$ has a fixed point. Repeating this argument m times, it follows that $T = c + K$ has a fixed point.

Finally, to verify the estimate (9), note first that the hypotheses given for Q ensure the existence of the minimal solution u_* of Eq. (10) if $|\bar{c} - c| \leqslant v$ (cf. [4]). Denoting $x = x(\cdot, c)$ and $\bar{x} = x(\cdot, \bar{c})$ we have

$$|\bar{x} - T\bar{x}| = |\bar{c} + K\bar{x} - c - K\bar{x}| = |\bar{c} - c|,$$

so that, if $|y - \bar{x}| \leqslant u_*$, then

$$\begin{aligned} |\bar{x} - Ty| &\leqslant |\bar{x} - T\bar{x}| + |T\bar{x} - Ty| \leqslant |\bar{c} - c| + Q\,|\bar{x} - y| \\ &\leqslant |\bar{c} - c| + Qu_* = u_*. \end{aligned}$$

Thus T maps the ball $B_{u_*}(\bar{x})$ into itself, whence

$$|\bar{x} - T^n \bar{x}| \leqslant u_* \qquad \text{for all } n = 0, 1, \ldots.$$

This implies estimate (9), as $n \to \infty$.

Remark. The results and the proof of Theorem 1 are slight modifications to those of Theorem 3.1 in [3] and Theorem 3.2 in [4].

We shall also apply the next local existence and uniqueness theorem whose proof is contained in that of Theorem 1.

THEOREM 2. *Let* $v, b \in C_+(\mathbb{R})$, $x_0 \in C(\mathbb{R})$ *and* $B_b(x_0) = \{x \in C(\mathbb{R}): |x - x_0| \leqslant b\}$. *Assume that* $K: C(\mathbb{R}) \to C(\mathbb{R})$ *satisfies*

$$|Kx - K\bar{x}| \leqslant Q(|x - \bar{x}|), \qquad x, \bar{x} \in B_b(x_0), \qquad |x - \bar{x}| \leqslant b, \tag{11}$$

where Q: $\{\theta \leqslant u \leqslant b\} \to \{\theta \leqslant u \leqslant b\}$ *satisfies the conditions of Theorem* 1. *If* $c \in C(\mathbb{R})$ *and* $|x_0 - c - Kx_0| \leqslant v$, *then the successive approximations* $x_{n+1} = c + Kx_n$, $n = 0, 1,\ldots$, *converge to the unique solution* $x(\cdot, c)$ *of Eq.* (6) *in* $B_b(x_0)$ *and*

$$|x(\cdot, c) - x_n| \leqslant Q^n b, \qquad n = 0, 1, 2,\ldots . \tag{12}$$

3. MAIN RESULTS

We now apply Theorems 1 and 2 in the study of the class of boundary value problem (1), (2). To this end we will require the existence of a continuous function $g: I \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the following conditions (cf. [5]).

(A_1) a positive real number p exists such that, for all $t \in I$,

$$p > \int_{t_0}^{t_1} k(t, s)\, g(s, p, p)\, ds,$$

(A_2) $g(t, u_1, v_1) \leqslant g(t, u_2, v_2)$ for all $t \in I$, $0 \leqslant u_1 \leqslant u_2 \leqslant p$, $0 \leqslant v_1 \leqslant v_2 \leqslant p$,

(A_3) the boundary value problem

$$\begin{aligned} Lu(t) &= g(t, u(t), u(\phi(t))), && t \in \dot{I}, \\ \alpha_0 u(t) - \alpha_1 u'(t) &= 0, && t \in I^-, \\ \beta_0 u(t) + \beta_1 u'(t) &= 0, && t \in I^+, \end{aligned}$$

has the unique solution $u = \theta$ in the segment $\{\theta \leqslant u \leqslant p\}$ of $C(\mathbb{R})$.

Using this function g we will also require either the Perron condition (A_4) for f, or its local version (A_5) given as follows.

(A_4) Inequality

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant g(t, |u_1 - u_2|, |v_1 - v_2|)$$

holds whenever $(t, u_i, v_i) \in I \times \mathbb{R} \times \mathbb{R}$, $i = 1, 2$ such that

$$|u_1 - u_2| \leqslant p \qquad \text{and} \qquad |v_1 - v_2| \leqslant p,$$

(A_5) (A_4) with the additional restrictions $|u_i - u_0| \leqslant p$ and $|v_i - v_0| \leqslant p$, $i = 1, 2$ for some fixed u_0 and $v_0 \in \mathbb{R}$.

THEOREM 3. *Let* (A_1)–(A_4) *hold. Then for any* $x_0 \in C(\mathbb{R})$ *the successive approximations*

$$x_{n+1} = c + Kx_n, \qquad n = 0, 1, \ldots,$$

where $c = z(\cdot, \alpha, \beta)$ *and* K *is defined by* (7), *converge to the unique solution* $x(\cdot, \alpha, \beta)$ *of the problem* (1), (2) *in* $C(\mathbb{R})$. *Moreover, there exist continuous and positive valued functions* $w_1 \colon I^- \to \mathbb{R}$ *and* $w_2 \colon I^+ \to \mathbb{R}$ *such that the conditions*

$$\begin{aligned} |\bar{\alpha}(t) - \alpha(t)| &\leqslant w_1(t) \qquad \textit{for} \quad t \in I^- \\ |\bar{\beta}(t) - \beta(t)| &\leqslant w_2(t) \qquad \textit{for} \quad t \in I^+ \end{aligned} \tag{13}$$

ensure

$$|x(\cdot, \bar{\alpha}, \bar{\beta}) - x(\cdot, \alpha, \beta)| \leqslant u_*, \tag{14}$$

where u_* *is the minimal solution of the boundary value problem*

$$\begin{aligned} Lu(t) &= g(t, u(t), u(\phi(t))), \qquad & t \in \dot{I}, \\ \alpha_0 u(t) - \alpha_1 u'(t) &= |\bar{\alpha}(t) - \alpha(t)|, \qquad & t \in I^-, \\ \beta_0 u(t) + \beta_1 u'(t) &= |\bar{\beta}(t) - \beta(t)|, \qquad & t \in I^+, \end{aligned} \tag{15}$$

and

$$\lim_{\bar{\alpha} \to \alpha,\, \bar{\beta} \to \beta} x(\cdot, \bar{\alpha}, \bar{\beta}) = x(\cdot, \alpha, \beta). \tag{16}$$

Proof. It follows from (A_4) that (8) holds with $b(t) \equiv p$ and Q defined by

$$Qu(t) = Fg(t, u(t), u(\phi(t))), \qquad t \in \mathbb{R}. \tag{17}$$

Conditions (A_1) and (A_2) ensure that Q is an increasing map of $\{\theta \leqslant u \leqslant b\}$ into itself. The order continuity and order compactness of Q follow from the Lebesgue dominated convergence theorem. Condition (A_3) implies that θ is the only fixed point of Q in $\{\theta \leqslant u \leqslant b\}$. Defining

$$\begin{aligned} v(t) &= p - Qb(t), \qquad & t \in \dot{I}, \\ &= p - Qb(t_0), \qquad & t \in I^-, \\ &= p - Qb(t_1), \qquad & t \in I^+, \end{aligned} \tag{18}$$

it follows from (A_1) that $v + Qb \leqslant b$. From (7) and (4) we see that K maps

$C(\mathbb{R})$ into the set of bounded functions of $C(\mathbb{R})$. Since $\min\{v(t): t \in \mathbb{R}\}$ is positive, then K is v-chainable. The existence and uniqueness conclusions of the theorem follow now from Theorem 1.

Denoting $\bar{c} = z(\cdot, \bar{\alpha}, \bar{\beta})$ and $\bar{v} = z(\cdot, |\bar{\alpha} - \alpha|, |\bar{\beta} - \beta|)$, it is a routine matter to prove that $|\bar{c} - c| \leqslant \bar{v}$ (cf. the Appendix). Thus existence of the minimal solution u_* of (15), i.e., that of $\bar{v} + Qu = u$, implies that the minimal solution of $|\bar{c} - c| + Qu = u$ exists and is majorized by u_* (cf. Lemma 2.1 of [4]). This, together with (9), implies (14), provided that u_* exists. Existence is guaranteed by (13), where $w_1: I^- \to \mathbb{R}_+$ and $w_2: I^+ \to \mathbb{R}_+$ can be any continuous functions for which $z(\cdot, w_1, w_2) \leqslant v$, because then $\bar{v} \leqslant z(\cdot, w_1, w_2) \leqslant v$ (again cf. Lemma 2.1 of [4]). The existence of such positive-valued functions w_1, w_2 is a consequence of the positivity of $\min\{v(t): t \in \mathbb{R}\}$ and the regularity and boundary conditions associated with L.

Finally, if (13) holds and if $\bar{\alpha} \to \alpha$ in $C(I^-)$ and $\bar{\beta} \to \beta$ in $C(I^+)$, then $\bar{v} \to \theta$ in $C_+(\mathbb{R})$, which implies that $u_* \to \theta$ in $C_+(\mathbb{R})$ (cf. [3, Lemma 2.4]). Thus, by (14), $x(\cdot, \bar{\alpha}, \bar{\beta}) \to x(\cdot, \alpha, \beta)$ in $C(\mathbb{R})$.

THEOREM 4. *Let* (A_1)–(A_3) *and* (A_5) *hold. If* $x_0 \in C(\mathbb{R})$ *with* $x_0(t) = u_0$, $x_0(\phi(t)) = v_0$, $t \in I$, *and if* $c = z(\cdot, \alpha, \beta)$ *satisfies* $|x_0 - c - Kx_0| \leqslant v$, *where* K *is defined by* (7) *and* v *by* (18), *then the successive approximations* $x_{n+1} = c + Kx_n$, $n = 0, 1, \ldots$ *converge to the unique solution* $x(\cdot, \alpha, \beta)$ *of problem* (1), (2) *in* $B_b(x_0)$. *For this convergence we have the estimate*

$$|x(\cdot, \alpha, \beta) - x_n| \leqslant Q^n b, \qquad n = 0, 1, \ldots . \tag{19}$$

Proof. It is easy to verify that the hypotheses of Theorem 2 are valid, thus implying the assertions.

Remarks 1. The existence and uniqueness result of Theorem 3 both generalizes that of Theorem 1 in [5], where the particular expression $Lx = -x''$ with Dirichlet boundary conditions is considered, and improves it in the sense that the first approximation x_0 can be any function of $C(\mathbb{R})$, and the uniqueness result is obtained in the whole of $C(\mathbb{R})$ instead of in some ball of $C(\mathbb{R})$.

2. The local condition (A_5), but not (A_4), is often satisfied for nonlinear functions f and g which are of convex type. Elliptic boundary value problems with convex nonlinearities generally have solutions which are not globally unique (cf. [7], for example). Therefore, in such cases Theorem 4 is more readily applicable than Theorem 3.

3. Condition (A_1) above can be weakened to the form

$(A_1)'$ There exists $b \in C_+(\mathbb{R})$ such that, for all $t \in I$,

$$b(t) > \int_{t_0}^{t_1} k(t, s)\, g(s, b(s)\, b(\phi(s)))\, ds.$$

An estimate for the absolute value of the solution to problem (1), (2) is now obtained provided there exists a continuous function $q: I \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that (cf. [5])

(D_1) $q(t, u_1, v_1) \leqslant q(t, u_2, v_2)$ whenever $t \in I$, $0 \leqslant u_1 \leqslant u_2$ and $0 \leqslant v_1 \leqslant v_2$,

(D_2) there exists a function $h \in C_+(\mathbb{R})$ such that, for all $t \in \mathbb{R}$, $z(t, |\alpha|, |\beta|) + Fq(t, h(t), h(\phi(t))) \leqslant h(t)$,

(D_3) $|f(t, u, v)| \leqslant q(t, |u|, |v|)$ for all $(t, u, v) \in I \times \mathbb{R} \times \mathbb{R}$.

THEOREM 5. *Let* (A_1)–(A_4) *and* (D_1)–(D_3) *hold. Then the unique solution x of problem* (1), (2) *satisfies*

$$|x| \leqslant w_* \tag{20}$$

and

$$|x - z(\cdot, \alpha, \beta)| \leqslant w_* - z(\cdot, |\alpha|, |\beta|), \tag{21}$$

where w_* *is the minimal solution of the boundary value problem*

$$\begin{aligned} Lw(t) &= q(t, w(t), w(\phi(t))), \qquad && t \in \dot{I}, \\ \alpha_0 w(t) - \alpha_1 w'(t) &= |\alpha(t)|, && t \in I^-, \\ \beta_0 w(t) + \beta_1 w'(t) &= |\beta(t)|, && t \in I^+. \end{aligned} \tag{22}$$

Proof. From (D_1) and (D_2) it follows that the equation

$$\Omega w(t) = Fq(t, w(t), w(\phi(t))), \qquad t \in \mathbb{R},$$

defines an increasing, order continuous and order compact mapping Ω from $C_+(\mathbb{R})$ into itself satisfying $z(\cdot, |\alpha|, |\beta|) + \Omega h \leqslant h$. Thus the minimal solution w_* of (22), i.e., the minimal solution of $z(\cdot, |\alpha|, |\beta|) + \Omega w = w$ exists (cf. [3, Lemma 2.2]). Since $|z(\cdot, \alpha, \beta)| \leqslant z(\cdot, |\alpha|, |\beta|)$, (cf. proof in the Appendix) then also the minimal solution $\bar{w}_*$ of $|z(\cdot, \alpha, \beta)| + \Omega w = w$ exists and $\bar{w}_* \leqslant w_*$. Since, by (D_3),

$$|Kx| \leqslant \Omega\, |x| \qquad \text{for all } x \in C(\mathbb{R}), \tag{23}$$

we obtain from Lemma 3.2 of [3] an estimate

$$|x| \leqslant \bar{w}_*$$

for the solution x of (1), (2), thus proving (20).

Since Ω is increasing, it follows from (20) and (23) that

$$|Kx| \leqslant \Omega\, |x| \leqslant \Omega w_*,$$

which implies (21).

4. Corollaries and Examples

To obtain more explicit results we prove

Corollary 1. *The results of Theorem* 3 (*resp. Theorem* 4) *hold if f and g are continuous and satisfy* (A_2), (A_4) (*resp.* (A_2), (A_5)) *and*

(A_0) *there exists $p > 0$ such that, for all $m \in (0, p]$,*

$$m > \int_{t_0}^{t_1} \bar{k}(s)\, g(s, m, m)\, ds, \qquad \textit{where} \quad \bar{k}(s) = \max_{t \in I}\{k(t, s)\},\ s \in I.$$

Proof. It suffices to show that g satisfies (A_1) and (A_3). Assumption (A_1) is a trivial consequence of (A_0). As for (A_3) we note that $u = \theta$ is, by (A_0), a solution of the boundary value problem given in (A_3). Assume now that u is any solution of this problem in the segment $\{\theta \leqslant u \leqslant p\}$ of $C_+(\mathbb{R})$. Then for all $t \in I$

$$u(t) = \int_{t_0}^{t_1} k(t, s)\, g(s, u(s), u(\phi(s)))\, ds.$$

Denoting $m = \max\{u(t): t \in I\}$, noting that $u(t) \leqslant m$ also for $t \in I^- \cup I^+$, and using (A_2) we obtain

$$u(t) \leqslant \int_{t_0}^{t_1} \bar{k}(s)\, g(s, m, m)\, ds$$

for all $t \in I$, so that

$$m \leqslant \int_{t_0}^{t_1} \bar{k}(s)\, g(s, m, m)\, ds.$$

This, together with (A_0), implies $m = 0$, i.e., $u = \theta$, which proves (A_3).

As an immediate consequence we obtain

Corollary 2. *Let $f: I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy for all $(t, u_i, v_i) \in I \times \mathbb{R} \times \mathbb{R}$, $i = 1, 2$,*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant \gamma(t)\, \omega(|u_1 - u_2|, |v_1 - v_2|), \tag{24}$$

where $\gamma \in C_+(I)$, $\omega: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing in both its arguments, and

$$\limsup_{m \to 0^+} \frac{\omega(m, m)}{m} < \left(\int_{t_0}^{t_1} \bar{k}(s)\, \gamma(s)\, ds \right)^{-1}. \tag{25}$$

Then the boundary value problem (1), (2) *has a unique solution which can be obtained by successive approximations. This holds in particular if f satisfies a Lipschitz condition*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant \gamma(t)(|u_1 - u_2| + |v_1 - v_2|), \tag{26}$$

where

$$\gamma \in C_+(I) \qquad \text{and} \qquad \int_{t_0}^{t_1} \bar{k}(s)\, \gamma(s)\, ds < 1/2.$$

Remark. The Lipschitz condition (26), without any restrictions on $\gamma \in C_+(I)$, does not ensure either existence or uniqueness of a solution to (1), (2). For example, the boundary value problem

$$-x''(t) = \lambda x(t), \qquad x(t) = \alpha(t), \qquad t \leqslant 0, \qquad x(t) = \beta(t), \qquad t \geqslant 1, \tag{27}$$

has for $\lambda = \pi^2$ no solution if $\alpha(0) \neq -\beta(1)$, and infinitely many solutions if $\alpha(0) = -\beta(1)$. On the other hand, if $\lambda < \pi^2$, then (27) has a unique solution. This follows also from Theorem 3 if $|\lambda| < 8$ and from the first part of Corollary 2 if $|\lambda| < 6$.

Since k is bounded as a continuous function, there follows from the latter part of Corollary 2 and from its local version (i.e., the result appropriate to Theorem 4):

COROLLARY 3. *If f is continuous and both $\partial f/\partial u$ and $\partial f/\partial v$ exist and are bounded, then the equation*

$$Lx(t) = \lambda f(t, x(t), x(\phi(t))), \tag{28}$$

with boundary conditions (2), *has a unique solution if $|\lambda|$ is small enough. If f, $\partial f/\partial u$ and $\partial f/\partial v$ are continuous and $f(t, 0, 0) \equiv 0$ then for each $p > 0$ there exists $\lambda_p > 0$ such that for $|\lambda| < \lambda_p$ the boundary value problem* (28), (2) *with $\alpha(t) \equiv 0$, $\beta(t) \equiv 0$ has $x = \theta$ as the only solution in $B_p(\theta)$.*

For example, the first part of Corollary 3 applies to the equation

$$Lx(t) = \lambda(\gamma_1(t)\, x(t)^{\delta_1} + \gamma_2(t)\, x(\phi(t))^{\delta_2}), \tag{29}$$

with $\gamma_i \in C_+(I)$ and boundary conditions (2), if $\delta_1 = \delta_2 = 1$, and the latter part of Corollary 3 if $\delta_1 > 1$ and $\delta_2 > 1$.

Remark. The above results can be extended naturally for problems involving more than one deviating argument, i.e., where Eq. (1) is replaced by

$$Lx(t) = f(t, x(t), x(\phi_1(t)), \ldots, x(\phi_\nu(t))).$$

Such nonlinear problems with two deviating arguments have appeared recently in relativistic electrodynamics (cf. [8] and references therein). Also, in the Debye–Hückel theory of electrolyte solutions, the approach in [9] leads to a problem involving a linear differential equation with both an advanced and a retarded argument. The following example deals with a slightly modified form of this problem (cf. [8]) and illustrates the simple way in which our results can be applied to a concrete problem.

EXAMPLE.

$$\begin{aligned} -x''(t) + \omega^2 x(t) &= \mu[h(t) + bx(t) + ax(t-c) + ax(t+c)], \\ & \qquad t \in (-\tau, \tau), \\ x(t) &= \alpha(t), \qquad t \in (-\infty, -\tau], \\ x(t) &= \beta(t), \qquad t \in [\tau, \infty), \end{aligned} \tag{30}$$

where $\mu, \omega, b \in \mathbb{R}$, $a, c \in \mathbb{R}_+$ and $h \in C(I)$, $I = [-\tau, \tau]$.

With $g(t, u, v, w) = |\mu|(|b|\, u + a(v + w))$, it is easy to prove (A_2) and (A_4).

Assumption (A_1) follows if $1 > \int_{-\tau}^{\tau} |\mu|(2a + |b|)\, k(t, s)\, ds$, where $k(t, s)$ satisfies

$$y(t) = \int_{-\tau}^{\tau} k(t, s)\, ds$$

and

$$\begin{aligned} -y''(t) + \omega^2 y(t) &= 1, \qquad t, s \in I, \\ y(-\tau) = y(\tau) &= 0. \end{aligned}$$

Thus

$$\frac{1}{|\mu|} > (2a + |b|) \sup_{t \in I} y(t),$$

i.e.,

$$\begin{aligned} \frac{1}{|\mu|} &> \frac{(2a + |b|)}{\omega^2} \left[1 - \frac{1}{\cosh \omega\tau}\right], \qquad \omega \neq 0, \\ &> (2a + |b|) \frac{\tau^2}{2}, \qquad \omega = 0, \end{aligned} \tag{31}$$

ensures that (A_1) holds.

Assuming that the homogenous boundary value problem (A_3) has only the trivial solution, Theorem 3 can now be applied to the problem (30) (cf. Theorem 6(i) of [8]). Furthermore the validity of condition (A_3) can be guaranteed by Corollary 1 provided

$$|\mu|(2a+|b|) < \left(\int_{-\tau}^{\tau} \bar{k}(s)\,ds\right)^{-1},$$

where $\bar{k}(s) = \max_{t\in I} k(t,s)$, $s \in I$. Hence the condition

$$\begin{aligned}
\frac{1}{|\mu|} &> (2a+|b|)\int_{-\tau}^{\tau} \bar{k}(s)\,ds \\
&= \frac{2a+|b|}{\omega^2}\left[\omega\tau\coth(2\omega\tau) - \frac{1}{2}\right], \qquad \omega \neq 0, \\
&= \frac{(4a+2|b|)\,\tau^2}{3}, \qquad \omega = 0,
\end{aligned}$$

ensures that the solution of (30) exists, is unique, is obtained by successive approximations and depends continuously on boundary functions α and β.

APPENDIX

The solution $c = z(\cdot, \alpha, \beta)$ of (1), (2) when f is the zero function has the explicit form:

$$\begin{aligned}
c(t) &= \zeta(t), \qquad t \in I, \\
&= e^{\alpha_0(t-t_0)/\alpha_1}\zeta(t_0) + \frac{1}{\alpha_1} e^{\alpha_0 t/\alpha_1}\int_t^{t_0} \alpha(s)\, e^{-\alpha_0 s/\alpha_1}\,ds, \\
&\qquad\qquad t \in I^-, \alpha_1 \neq 0, \\
&= \frac{1}{\alpha_0}\alpha(t), \qquad t \in I^-, \alpha_1 = 0, \\
&= e^{-\beta_0(t-t_1)/\beta_1}\zeta(t_1) + \frac{1}{\beta_1} e^{-\beta_0 t/\beta_1}\int_{t_1}^{t} \beta(s)\, e^{\beta_0 s/\beta_1}\,ds, \\
&\qquad\qquad t \in I^+, \beta_1 \neq 0, \\
&= \frac{1}{\beta_0}\beta(t), \qquad t \in I^+, \beta_1 = 0,
\end{aligned}$$

where ζ is the unique solution of the boundary value problem

$$L\zeta(t) = 0, \qquad t \in \dot{I},$$
$$\alpha_0 \zeta(t_0) - \alpha_1 \zeta'(t_0) = \alpha(t_0),$$
$$\beta_0 \zeta(t_1) + \beta_1 \zeta'(t_1) = \beta(t_1).$$

Using the notation of Theorem 3, and denoting $y = \bar{v} - (c - \bar{c})$, we obtain

$$Ly(t) = 0, \qquad t \in \dot{I}$$

with

$$\alpha_0 y(t_0) - \alpha_1 y'(t_0) \geqslant 0, \qquad \beta_0 y(t_1) + \beta_1 y'(t_1) \geqslant 0.$$

It follows from the maximum principle that $y(t) \geqslant 0$ on I.

For $t \in I^- \cup I^+$, the above expressions all give $y(t) \geqslant 0$. Consequently $\bar{v}(t) \geqslant (c - \bar{c})(t)$ for all $t \in \mathbb{R}$. Similarly, we can prove that $\bar{v}(t) \geqslant (\bar{c} - c)(t)$ and thus $|\bar{c} - c| \leqslant \bar{v}$.

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